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COMMENT

Tricritical point in a random Potts model

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Abstract. An approximate mapping between a random Potts model with a different number (q and p) of Potts spin states and the Ising model is used to locate the concentration at which the change of the order of the phase transition occurs. Results are obtained for $d=2$, $q=3$, $p=5$, 6, 7, and $d=3$, $q=2$, $p=3$.

A random model which consists of two types (p and q states) of Potts spins was first considered by Miyazima [1]. Its particular interest derives from the fact that the crossover between the critical behaviour of the q-state model and the critical behaviour of the p-state Potts model can be obtained as the concentration of both types of spins is varied.

Mean-field approximation predicts a first-order phase transition for the Potts model with $q>2$, independent of the dimensionality [2]. Accordingly, Miyazima's [1] use of this type of approximation leads to the location, for $q=3, p=6$, of a line of first-order phase transitions. These predictions are known to be incorrect for $q \leqslant q_c(d)$, when the q-state Potts model undergoes a second-order phase transition.

An improvement on Miyazima's calculation was recently obtained [3] by use of mean-field renormalisation group (MFRG) [4] techniques. This method is however only entirely justified in the case of second-order phase transitions.

In two dimensions, $q_c = 4$ [2], therefore, the transition is continuous for the three-state Potts model and discontinuous for the six-state Potts model, whereas for the mixed Potts model there must be a certain concentration, x_c, for which the phase transition changes order. However, the limitations of the MFRG method in what concerns a computation of the free energy prevent its use for the determination of x_c.

In this comment, we use an approximate mapping of the mixed Potts model onto an Ising model, a method used by Vause and Walker [5] for the determination of $q_c(d)$. Such mapping is possible for a two-spin cluster due to the fact that for both models the nearest-neighbour energy spectrum consists of two levels.

In the case of the mixed Potts model we study, the statistical variable at each lattice site i can be written as

$$\mathcal{S}_i = \eta_i \sigma_i + (1-\eta_i)\bar{\sigma}_i$$

where σ_i takes on p values $\sigma_i = 1, \ldots, p$, $\bar{\sigma}_i$ takes on q values $\bar{\sigma}_i = 1, \ldots, q$, and η_i is a random variable which takes the value 1(0) with probability x $(1-x)$ if site i is occupied by a $p(q)$-component Potts spin.

The two-spin cluster approximation is then

$$\exp(K_0 + h_1(S_i+S_j) + K_1 S_i S_j] = \sum_{\mathcal{S}_i \mathcal{S}_j} \overline{P(S_i S_j | \mathcal{S}_i \mathcal{S}_j)} \exp(K_P \delta_{\mathcal{S}_i, \mathcal{S}_j}) \qquad (S_i, S_j = \pm 1) \quad (1)$$

where K_0 is a spin-independent constant, K_P is the Potts coupling, h_I and K_I are the effective Ising external field and coupling, respectively, and $\overline{P(S_iS_j|\mathscr{S}_i\mathscr{S}_j)}$ is a configurational average of an appropriate projection operator

$$P(S_iS_j|\mathscr{S}_i\mathscr{S}_j)=\tfrac{1}{4}[1-S_i(2\delta_{\mathscr{S}_{i},1}-1)(2\delta_{\mathscr{S}_i,\mathscr{S}_j}-1)][1-S_j(2\delta_{\mathscr{S}_{i},1}-1)(2\delta_{\mathscr{S}_i,\mathscr{S}_j}-1)].$$

The evaluation of the statistical sum in (1) for the Ising two-spin configurations ($S_i=+1$, $S_j=+1$), ($S_i=-1$, $S_j=-1$) and ($S_i=+1$, $S_j=-1$) leads to

$$\begin{aligned} e^{4h_I}&=\frac{x^2(p-1)+(1-x^2)(q-1)}{1+\{x(p-2)[x(p-q)+q-1]+(1-x)(q-1)[x(p-q)+q-2]\}\,e^{-K_P}} \\ e^{2K_0}&=[x(p-1)+(1-x)(q-1)]\{e^{K_P}+x(p-2)[x(p-q)+q-1] \\ &\qquad +(1-x)(q-1)[x(p-q)+q-2)]\}\,e^{2h_I} \\ K_I&=K_0-\ln[x(p-1)+(1-x)(q-1)]. \end{aligned} \tag{2}$$

The transition is first order for $h_I=0$, $K_I>K_{IC}(d)$ (where $K_{IC}(d)$ is the critical ferromagnetic Ising coupling in d dimensions), i.e. when

$$\tfrac{1}{2}\ln\left(\frac{\{x(p-2)[x(p-q)+q-1]+(1-x)(q-1)[x(p-q)+q-2]\}\times[x^2(p-1)+(1-x^2)(q-1)]}{[x(p-1)+(1-x)(q-1)][x^2(p-1)+(1-x^2)(q-1)-1]}\right)>K_{IC}(d). \tag{3}$$

The concentration x_c for which a change of order occurs corresponds to the inequality (3) being satisfied as an equality.

For $d=2$, we use the Onsager exact result $K_{IC}(2)=\frac{1}{2}\ln(1+\sqrt{2})$. We set $q=3$ and determine x_c for $p=5, 6, 7$; we obtain, respectively, $x_c=0.067$, 0.045, 0.034.

For $d=3$, the best estimates give $q_c(3)=0.26$ [6] and $K_{IC}(3)=0.222$ [7]. For a mixed Potts model with $q=2$, $p=3$, we find a tricritical point at $x_c=0.93$.

As far as we know it is the first time that tricritical behaviour has been investigated in a model with mixed Potts spins; the tricritical point is reached here by continuous variation of the relative concentration of two types of Potts spins. The changeover from continuous to first-order phase transition, which was first predicted by Baxter [8] based on the equivalence of two-dimensional Potts and certain ice-rule vertex models has also been studied by Nienhuis *et al* [9] by an enlargement of the parameter space to include dilution. In both approaches q is taken as a parameter which can vary continuously, whereas with this model we study the more physical situation where q and p are fixed integers and the concentration varies continuously.

We think that the consideration of larger clusters will improve the numerical accuracy of the present results as it is also expected to improve the estimate obtained by MFRG for the location of the critical line.

The study of critical exponents in the present model would be of much interest, enabling one to investigate the crossover between critical and tricritical behaviour as the concentration approaches x_c.

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